Uniform asymptotic stability of solutions of fractional functional differential equations *

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Abstract In this paper, some global existence and uniform asymptotic stability results for fractional functional differential equations are proved. It is worthy mentioning that when $\alpha = 1$ the initial value problem (1.1) reduces to a classical dissipative differential equation with delays in [4].

Keywords: Functional differential equation; Fractional derivative; Asymptotic stability; Global existence.

1 Introduction

Consider the initial value problem (IVP for short) of the following fractional functional differential equation:

$$\begin{cases} D^{\alpha}\left[y(t)e^{\beta t}\right] = f(t, y_t)e^{\beta t}, \quad t \in [t_0, \infty),\, t_0 \geqslant 0,\, 0 < \alpha < 1, \\ y(t) = \phi(t), \quad t_0 - h \leqslant t \leqslant t_0, \end{cases} \tag{1.1}$$

where D^{α} is the Caputo fractional derivative, $\beta > 0$, $f : J \times C([-h, 0], \mathbb{R}) \to \mathbb{R}$, where $J = [t_0, \infty)$, is a given function satisfying some assumptions that will be specified later, $h > 0$, and $\phi \in C([t_0 - h, t_0], \mathbb{R})$. If $y \in C([t_0 - h, \infty), \mathbb{R})$, then for any $t \in [t_0, \infty)$, define y_t by

$$y_t(\theta) = y(t + \theta), \quad \theta \in [-h, 0].$$

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The study of retarded differential equations is an important area of applied mathematics due to physical reasons, non-instant transmission phenomena, memory processes, and specially biological motivations (see, e.g., [4, 13, 16, 17]). Fractional differential equations have attracted much attention recently (see, for example, [2, 3, 11, 12, 15, 18, 19] and the references cited therein for the applications in various sciences such as physics, mechanics, chemistry, engineering, etc).

Some attractivity results for fractional functional differential equations and nonlinear functional integral equations are obtained by using the fixed point theory; see [5, 6, 8, 9, 10] and references therein. Global asymptotic stability of solutions of a functional integral equation is discussed in [1], however there is no work on uniform asymptotic stability of solutions of fractional functional differential equation. It is our intention here to show the global existence and uniform asymptotic stability of the fractional functional differential equation (1.1).

We organize the paper as follows. In Section 2, we recall some necessary concepts and results. In Section 3 we give the global existence and uniform asymptotic stability of fractional functional differential equations. Finally, two examples are given to illustrate our main results.

2 Preliminaries

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper.

We consider $BC := BC([t_0 - h, \infty), \mathbb{R})$ the Banach space of all bounded and continuous functions from $[t_0 - h, \infty)$ into $\mathbb{R}$ with the norm

$$\|y\|_\infty := \sup \{|y(t)| \ : \ t \in [t_0 - h, \infty)\}.$$

Let $\|y_t\| = \sup_{-h \leqslant \theta \leqslant 0} |y(t+\theta)|$ for $t \in J$.

Throughout this paper, we always assume that $f(t, x_t)$ satisfies the following condition:

(H_0) $f(t, x_t)$ is Lebesgue measurable with respect to t on $[t_0, \infty)$, and $f(t, \varphi)$ is continuous with respect to φ on $C([-h, 0], \mathbb{R})$.

By condition (H_0) and the technique used in [11], we get the equivalent form of IVP (1.1) as:

$$y(t) = \begin{cases} y(t_0)e^{-\beta(t-t_0)} + \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_{t_0}^{t} (t-s)^{\alpha-1} e^{-\beta(t-s)} f(s, y_s) ds, \quad t \geqslant t_0, \\ \phi(t), \quad t \in [t_0 - h, t_0]. \end{cases} \tag{2.1}$$

Definition 2.1. *We say that solutions of IVP* (1.1) *are uniformly asymptotically stable if for any bounded subset B of $C([-h, 0], \mathbb{R})$ and $\varepsilon > 0$, there exists a $T > 0$ such that*

$$|y(t, t_0, \phi) - x(t, t_0, \psi)| \leqslant \varepsilon \quad \text{for all } t \geqslant T \text{ and } \phi, \psi \in B.$$

We recall the following generalization of Gronwall's lemma for singular kernels [14], which will be used in the sequel.

Lemma 2.2. *Let $v : [t_0, b] \to [0, +\infty)$ be a real function and $w(\cdot)$ is a nonnegative, locally integrable function on $[t_0, b]$ and there are constants $a > 0$ and $0 < \alpha < 1$ such that*

$$v(t) \leqslant w(t) + a \int_{t_0}^{t} \frac{v(s)}{(t-s)^{\alpha}} ds.$$

Then there exists a constant $K = K(\alpha)$ such that

$$v(t) \leqslant w(t) + Ka \int_{t_0}^{t} \frac{w(s)}{(t-s)^{\alpha}} ds,$$

for every $t \in [t_0, b]$.

Theorem 2.3 (Leray-Schauder Fixed Point Theorem). *Let P be a continuous and compact mapping of a Banach space X into itself, such that the set*

$$\{x \in X \ : \ x = \lambda P x \text{ for some } 0 \leqslant \lambda \leqslant 1\}$$

is bounded. Then P has a fixed point.

3 FDEs of fractional order

In this section, we will investigate the IVP (1.1). Our first global existence and uniform asymptotic stability result for the IVP (1.1) is based on the Banach contradiction principle and Lemma 2.2.

Theorem 3.1. *Assume that $f(t, y_t)$ satisfies conditions (H_0) and*

(H_1) *there exists $l > 0$ such that*

$$|f(t, u_t) - f(t, v_t)| \leqslant l\|u_t - v_t\| \tag{3.1}$$

for $t \in J$ and every $u_t, v_t \in C([-h, 0], \mathbb{R})$. Moreover, the function $t \mapsto f(t, 0)$ is bounded with $f_0 = \sup_{t \geqslant t_0} |f(t, 0)|$.

If

$$l\left(\frac{(t_0+h)^{\alpha-1}e^{-\beta(t_0+h)}}{\beta\Gamma(\alpha)} + \frac{(t_0+h)^{\alpha}}{\Gamma(\alpha+1)}\right) < 1, \tag{3.2}$$

then the IVP (1.1) has a unique solution in the space BC. Moreover, solutions of IVP (1.1) are uniformly asymptotically stable.

Proof. We divide the proof into two steps.

Step1. We define the operator $P : C([t_0 - h, \infty), \mathbb{R}) \to C([t_0 - h, \infty), \mathbb{R})$ by

$$(Py)(t) = \begin{cases} y(t_0)e^{-\beta(t-t_0)} + \dfrac{1}{\Gamma(\alpha)}\displaystyle\int_{t_0}^{t}(t-s)^{\alpha-1}e^{-\beta(t-s)}f(s, y_s)ds, \quad t \geqslant t_0, \\ \phi(t), \quad t \in [t_0 - h, t_0]. \end{cases} \tag{3.3}$$

The operator P maps BC into itself. Indeed for each $y \in BC$, and for each $t \geqslant 2t_0 + h$, it follows

from (H_1) that

$$\begin{aligned}
|(Py)(t)| &\leqslant |y(t_0)|e^{-\beta(t-t_0)} + \frac{1}{\Gamma(\alpha)}\int\limits_{t_0}^{t}(t-s)^{\alpha-1}e^{-\beta(t-s)}(f_0+l\|y_s\|)ds \\
&\leqslant \|\phi\|e^{-\beta(t-t_0)} + \frac{f_0+l\|y\|_\infty}{\Gamma(\alpha)}\left(\int\limits_{t_0}^{t-(t_0+h)}(t_0+h)^{\alpha-1}e^{-\beta(t-s)}ds + \int\limits_{t-(t_0+h)}^{t}(t-s)^{\alpha-1}ds\right) \\
&\leqslant \|\phi\| + (f_0+l\|y\|_\infty)\left(\frac{(t_0+h)^{\alpha-1}e^{-\beta(t_0+h)}}{\beta\Gamma(\alpha)} + \frac{(t_0+h)^{\alpha}}{\Gamma(\alpha+1)}\right),
\end{aligned}$$

for each $t \in [t_0, 2t_0+h]$ we have

$$|(Py)(t)| \leqslant \|\phi\| + \frac{(f_0+l\|y\|_\infty)(t_0+h)^\alpha}{\Gamma(\alpha+1)},$$

and consequently $P(y) \in BC$.

Since $BC := BC([t_0-h,\infty),\mathbb{R})$ is a Banach space with norm $\|\cdot\|_\infty$, we shall show that $P : BC \to BC$ is a contraction map. Let $y_1, y_2 \in BC$. Then we have for each $t \geqslant t_0$,

$$\begin{aligned}
|(Py_1)(t)-(Py_2)(t)| &\leqslant \frac{1}{\Gamma(\alpha)}\int\limits_{t_0}^{t}(t-s)^{\alpha-1}e^{-\beta(t-s)}|f(s,y_{1s})-f(s,y_{2s})|ds \\
&\leqslant \frac{l}{\Gamma(\alpha)}\int\limits_{t_0}^{t}(t-s)^{\alpha-1}e^{-\beta(t-s)}\|y_{1s}-y_{2s}\|ds.
\end{aligned} \tag{3.4}$$

Therefore for any $t \geqslant 2t_0+h$,

$$\begin{aligned}
&|(Py_1)(t)-(Py_2)(t)| \\
&\leqslant \frac{l}{\Gamma(\alpha)}\|y_1(\cdot)-y_2(\cdot)\|_\infty\left(\int\limits_{t_0}^{t-(t_0+h)}(t_0+h)^{\alpha-1}e^{-\beta(t-s)}ds + \int\limits_{t-(t_0+h)}^{t}(t-s)^{\alpha-1}ds\right) \\
&\leqslant l\left(\frac{(t_0+h)^{\alpha-1}e^{-\beta(t_0+h)}}{\beta\Gamma(\alpha)} + \frac{(t_0+h)^\alpha}{\Gamma(\alpha+1)}\right)\|y_1(\cdot)-y_2(\cdot)\|_\infty,
\end{aligned} \tag{3.5}$$

and for $t_0-h \leqslant t \leqslant 2t_0+h$,

$$\begin{aligned}
|(Py_1)(t)-(Py_2)(t)| &\leqslant \frac{l}{\Gamma(\alpha)}\|y_1(\cdot)-y_2(\cdot)\|_\infty\int\limits_{t_0}^{t}(t-s)^{\alpha-1}ds \\
&\leqslant \frac{l(t_0+h)^\alpha}{\Gamma(\alpha+1)}\|y_1(\cdot)-y_2(\cdot)\|_\infty,
\end{aligned} \tag{3.6}$$

and thus

$$\begin{aligned}
&\|(Py_1)(\cdot)-(Py_2)(\cdot)\|_\infty \\
&\leqslant l\left(\frac{(t_0+h)^{\alpha-1}e^{-\beta(t_0+h)}}{\beta\Gamma(\alpha)}+\frac{(t_0+h)^\alpha}{\Gamma(\alpha+1)}\right)\|y_1(\cdot)-y_2(\cdot)\|_\infty.
\end{aligned}\tag{3.7}$$

Hence, (3.2) and (3.7) imply that the operator P is a contraction. Therefore, P has a unique fixed point by Banach's contraction principle.

Step2. For any two solutions $x=x(t)$ and $y=y(t)$ of IVP (1.1) corresponding to initial values ψ and ϕ, by (2.1) we can deduce that for all $t\geqslant t_0+h$ and all $\theta\in[-h,0]$,

$$\begin{aligned}
&|x(t+\theta)-y(t+\theta)|\leqslant|x(t_0)-y(t_0)|e^{-\beta(t+\theta-t_0)} \\
&\quad+\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t+\theta}(t+\theta-s)^{\alpha-1}e^{-\beta(t+\theta-s)}|f(s,x_s)-f(s,y_s)|ds \\
&\leqslant|x(t_0)-y(t_0)|e^{-\beta(t+\theta-t_0)}+\frac{l}{\Gamma(\alpha)}\int_{t_0}^{t+\theta}(t+\theta-s)^{\alpha-1}e^{-\beta(t+\theta-s)}\|x_s-y_s\|ds.
\end{aligned}\tag{3.8}$$

Then, it follows that

$$e^{\beta t}\|x_t-y_t\|\leqslant|x(t_0)-y(t_0)|e^{\beta(h+t_0)}+\frac{le^{\beta h}}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}e^{\beta s}\|x_s-y_s\|ds.\tag{3.9}$$

Let $w(t)=e^{\beta t}\|x_t-y_t\|$. Then we have

$$w(t)\leqslant|x(t_0)-y(t_0)|e^{\beta(h+t_0)}+\frac{le^{\beta h}}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}w(s)ds.\tag{3.10}$$

Applying Lemma 2.2, one can see that there exists a constant K such that

$$\begin{aligned}
w(t)\ &\leqslant\ |x(t_0)-y(t_0)|e^{\beta(h+t_0)}+\frac{Kle^{\beta h}}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}|x(t_0)-y(t_0)|e^{\beta(h+t_0)}ds \\
&\leqslant\ |x(t_0)-y(t_0)|e^{\beta(h+t_0)}\left(1+\frac{Kle^{\beta h}}{\Gamma(\alpha+1)}(t-t_0)^\alpha\right).
\end{aligned}\tag{3.11}$$

Hence we obtain

$$e^{\beta t}\|x_t-y_t\|=w(t)\leqslant|x(t_0)-y(t_0)|e^{\beta(h+t_0)}\left(1+\frac{Kle^{\beta h}}{\Gamma(\alpha+1)}(t-t_0)^\alpha\right),$$

and thus for all $t\geqslant t_0+h$,

$$|x(t)-y(t)|\leqslant|x(t_0)-y(t_0)|e^{-\beta(t-h-t_0)}\left(1+\frac{Kle^{\beta h}}{\Gamma(\alpha+1)}(t-t_0)^\alpha\right),$$

which implies that the solutions of IVP (1.1) are uniformly asymptotically stable. $\square$

Now we give global existence and uniform asymptotic stability results based on the nonlinear alternative of Leray-Schauder type.

Theorem 3.2. *Assume that the following hypotheses hold:*

(H_2) *f is a continuous function;*

(H_3) *there exist positive functions k_1, $k_2 \in BC([t_0, \infty), \mathbb{R}_+)$ such that*

$$|f(t, u_t)| \leqslant k_1(t) + k_2(t)\|u_t\|$$

for $t \in J$ and every $u_t \in C([-h, 0], \mathbb{R})$;

(H_4) *moreover, assume that $K_1 = \sup_{t \geqslant t_0} k_1(t)$, $K_2 = \sup_{t \geqslant t_0} k_2(t)$,*

$$\lim_{t \to \infty} \int_{t_0}^{t} (t-s)^{\alpha-1} e^{-\beta(t-s)} k_1(s) ds = 0,$$

and

$$\lim_{t \to \infty} \int_{t_0}^{t} (t-s)^{\alpha-1} e^{-\beta(t-s)} k_2(s) ds = 0.$$

Then the IVP (1.1) admits a solution in the space BC. Moreover, solutions of IVP (1.1) are uniformly asymptotically stable.

Proof. Let $P : C([t_0 - h, \infty), \mathbb{R}) \to C([t_0 - h, \infty), \mathbb{R})$ be defined as in (3.3). First we show that P maps BC into itself. Indeed, the map $P(y)$ is continuous on $[t_0 - h, +\infty)$ for each $y \in BC$, and for each $t \geqslant 2t_0 + h$, (H_2) implies that

$$\begin{aligned}
|(Py)(t)| &\leqslant |y(t_0)|e^{-\beta(t-t_0)} + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} e^{-\beta(t-s)} (K_1 + K_2\|y_s\|) ds \\
&\leqslant \|\phi\| e^{-\beta(t-t_0)} + \frac{K_1 + K_2\|y\|_\infty}{\Gamma(\alpha)} \left(\int_{t_0}^{t-(t_0+h)} (t_0+h)^{\alpha-1} e^{-\beta(t-s)} ds + \int_{t-(t_0+h)}^{t} (t-s)^{\alpha-1} ds \right) \\
&\leqslant \|\phi\| + (K_1 + K_2\|y\|_\infty) \left(\frac{(t_0+h)^{\alpha-1} e^{-\beta(t_0+h)}}{\beta\Gamma(\alpha)} + \frac{(t_0+h)^\alpha}{\Gamma(\alpha+1)} \right),
\end{aligned} \tag{3.12}$$

for each $t \in [t_0, 2t_0 + h]$ we have

$$|(Py)(t)| \leqslant \|\phi\| + \frac{(K_1 + K_2\|y\|_\infty)(t_0 + h)^\alpha}{\Gamma(\alpha + 1)}, \tag{3.13}$$

and for any $t \in [t_0 - h, t_0]$,

$$|(Py)(t)| \leqslant \|\phi\|.$$

Thus,

$$\|P(y)\|_\infty \leqslant \|\phi\| + (K_1 + K_2\|y\|_\infty)\left(\frac{(t_0 + h)^{\alpha-1}e^{-\beta(t_0+h)}}{\beta\Gamma(\alpha)} + \frac{(t_0 + h)^\alpha}{\Gamma(\alpha + 1)}\right),$$

and consequently $P(y) \in BC$.

Next, we show that the operator P is continuous and completely continuous, and there exists an open set $U \subset BC$ with $y \neq \lambda P(y)$ for $\lambda \in (0, 1)$ and $y \in \partial U$.

Step 1. P is continuous.

Let $\{y_n\}$ be a sequence such that $y_n \to y$ in BC. Then there exist $R > 0$ and $N > 0$ such that

$$\|y_n\|_\infty + \|y\|_\infty < R, \quad \forall n \geqslant N. \tag{3.14}$$

Let $\varepsilon > 0$ be given. Since (H_4) holds, there is a real number $T > 0$ such that

$$\frac{2}{\Gamma(\alpha)}\int_{t_0}^{t}(t - s)^{\alpha-1}e^{-\beta(t-s)}(k_1(s) + k_2(s)R)ds < \varepsilon \tag{3.15}$$

for all $t \geqslant T$. Now we consider the following two cases.

Case 1: if $t \geqslant T$, then it follows from (H_3) and (3.14)-(3.15) that for n sufficiently large

$$\begin{aligned}|Py_n(t) - Py(t)| &\leqslant |y_n(t_0) - y(t_0)|e^{-\beta(t-t_0)} + \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t - s)^{\alpha-1}e^{-\beta(t-s)}|f(s, y_{ns}) - f(s, y_s)|ds\\ &\leqslant |y_n(t_0) - y(t_0)| + \frac{2}{\Gamma(\alpha)}\int_{t_0}^{t}(t - s)^{\alpha-1}e^{-\beta(t-s)}(k_1(s) + k_2(s)R)ds < 2\varepsilon.\end{aligned} \tag{3.16}$$

Case 2: if $t_0 \leqslant t \leqslant T$, since f is a continuous function, one has

$$\begin{aligned} |Py_n(t) - Py(t)| &\leqslant |y_n(t_0) - y(t_0)| + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} e^{-\beta(t-s)} |f(s, y_{ns}) - f(s, y_s)| ds \\ &\leqslant |y_n(t_0) - y(t_0)| + \frac{(T-t_0)^\alpha}{\Gamma(\alpha+1)} \sup_{s \in [t_0, T]} |f(s, y_{ns}) - f(s, y_s)|. \end{aligned} \tag{3.17}$$

Note that $y_n \to y$ in BC. Hence (3.16) and (3.17) imply that

$$\|P(y_n) - P(y)\|_\infty \to 0 \quad \text{ as } n \to \infty.$$

Step 2. P maps bounded sets into bounded sets in BC.

Indeed, it is enough to show that for any $\eta > 0$, there exists a positive constant ℓ such that for each $y \in B_\eta = \{y \in BC : ||y||_\infty \leqslant \eta\}$ one has $\|P(y)\|_\infty \leqslant \ell$. Let $y \in B_\eta$. Then we have for each $t \geqslant 2t_0 + h$,

$$\begin{aligned} |(Py)(t)| &\leqslant |y(t_0)| e^{-\beta(t-t_0)} + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} e^{-\beta(t-s)} |f(s, y_s)| ds \\ &\leqslant \eta + \frac{K_1 + K_2\|y\|_\infty}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} e^{-\beta(t-s)} ds \\ &\leqslant \eta + (K_1 + K_2\eta) \left(\frac{(t_0+h)^{\alpha-1} e^{-\beta(t_0+h)}}{\beta\Gamma(\alpha)} + \frac{(t_0+h)^\alpha}{\Gamma(\alpha+1)} \right) =: \ell, \end{aligned}$$

and for each t with $t_0 \leqslant t \leqslant 2t_0 + h$,

$$|(Py)(t)| \leqslant \eta + (K_1 + K_2\eta) \frac{(t_0+h)^\alpha}{\Gamma(\alpha+1)}.$$

Hence $||P(y)||_\infty \leqslant \ell$.

Step 3. P maps bounded sets into equicontinuous sets on every compact subset $[t_0 - h, b]$ of $[t_0 - h, \infty)$.

Let $t_1, t_2 \in [t_0, b]$, $t_1 < t_2$, and let B_η be a bounded set of BC as in Step 2. Let $y \in B_\eta$. Then

we have

$$
\begin{aligned}
|(Py)(t_2)-(Py)(t_1)| &\leqslant |y(t_0)e^{-\beta(t_2-t_0)}-y(t_0)e^{-\beta(t_1-t_0)}| \\
&+\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t_1}\left|\left((t_2-s)^{\alpha-1}e^{-\beta(t_2-s)}-(t_1-s)^{\alpha-1}e^{-\beta(t_1-s)}\right)f(s,y_s)\right|ds \\
&+\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}\left|(t_2-s)^{\alpha-1}e^{-\beta(t_2-s)}f(s,y_s)\right|ds \\
&\leqslant |y(t_0)|e^{\beta t_0}|e^{-\beta t_2}-e^{-\beta t_1}|+\frac{K_1+K_2\eta}{\Gamma(\alpha+1)}(t_2-t_1)^{\alpha} \\
&+\frac{K_1+K_2\eta}{\Gamma(\alpha)}\int_{t_0}^{t_1}\left((t_1-s)^{\alpha-1}e^{-\beta(t_1-s)}-(t_2-s)^{\alpha-1}e^{-\beta(t_2-s)}\right)ds.
\end{aligned}
\tag{3.18}
$$

Observing that

$$
\begin{aligned}
&\frac{K_1+K_2\eta}{\Gamma(\alpha)}\int_{t_0}^{t_1}\left((t_1-s)^{\alpha-1}e^{-\beta(t_1-s)}-(t_2-s)^{\alpha-1}e^{-\beta(t_1-s)}\right)ds \\
&\leqslant \frac{K_1+K_2\eta}{\Gamma(\alpha)}\int_{t_0}^{t_1}\left((t_1-s)^{\alpha-1}-(t_2-s)^{\alpha-1}\right)ds \\
&\leqslant \frac{K_1+K_2\eta}{\Gamma(\alpha+1)}\left((t_1-t_0)^{\alpha}-(t_2-t_0)^{\alpha}+(t_2-t_1)^{\alpha}\right) \\
&\leqslant \frac{K_1+K_2\eta}{\Gamma(\alpha+1)}(t_2-t_1)^{\alpha},
\end{aligned}
\tag{3.19}
$$

and from Taylor's theorem, we obtain

$$
\begin{aligned}
&\frac{K_1+K_2\eta}{\Gamma(\alpha)}\int_{t_0}^{t_1}\left((t_2-s)^{\alpha-1}e^{-\beta(t_1-s)}-(t_2-s)^{\alpha-1}e^{-\beta(t_2-s)}\right)ds \\
&\leqslant \frac{K_1+K_2\eta}{\Gamma(\alpha)}(t_2-t_1)^{\alpha-1}\int_{t_0}^{t_1}\left(e^{-\beta(t_1-s)}-e^{-\beta(t_2-s)}\right)ds \\
&\leqslant \frac{K_1+K_2\eta}{\beta\Gamma(\alpha)}(t_2-t_1)^{\alpha-1}\left(1-e^{-\beta(t_2-t_1)}\right) \\
&= \frac{K_1+K_2\eta}{\Gamma(\alpha)}\left((t_2-t_1)^{\alpha}+\frac{o(t_2-t_1)}{t_2-t_1}(t_2-t_1)^{\alpha}\right),
\end{aligned}
\tag{3.20}
$$

where $\lim_{t_2-t_1\to 0}\frac{o(t_2-t_1)}{t_2-t_1}=0$. By (3.18)-(3.20), we can conclude that

$$
\begin{aligned}
|(Py)(t_2)-(Py)(t_1)| &\leqslant \eta e^{\beta t_0}|e^{-\beta t_2}-e^{-\beta t_1}| \\
&+\frac{2(K_1+K_2\eta)}{\Gamma(\alpha+1)}(t_2-t_1)^{\alpha}+\frac{K_1+K_2\eta}{\Gamma(\alpha)}\left((t_2-t_1)^{\alpha}+\frac{o(t_2-t_1)}{t_2-t_1}(t_2-t_1)^{\alpha}\right).
\end{aligned}
$$

As $t_1 \to t_2$ the right-hand side of the above inequality tends to zero. The equicontinuity for the cases $t_1 < t_2 \leqslant t_0$ and $t_1 \leqslant t_0 \leqslant t_2$ is obvious.

Step 4. P maps bounded sets into equiconvergent sets.

Let $y \in B_\eta$. Then

$$|(Py)(t)| \leqslant |y(t_0)|e^{-\beta(t-t_0)} + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} e^{-\beta(t-s)} |f(s, y_s)| ds$$
$$\leqslant \eta e^{-\beta(t-t_0)} + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} e^{-\beta(t-s)} \left(k_1(s) + k_2(s)\eta\right) ds.$$

Therefore (H_4) implies that $|(Py)(t)|$ uniformly (w.r.t $y \in B(\eta)$) converges to 0 as $t \to \infty$. As a consequence of Steps 1-4, we can conclude that $P : BC \to BC$ is continuous and completely continuous.

Step 5 (A priori bounds). We now show there exists an open set $U \subseteq BC$ with $y \neq \lambda P(y)$ for $\lambda \in (0, 1)$ and $y \in \partial U$.

Let $y \in BC$ and $y = \lambda P(y)$ for some $0 < \lambda < 1$. Then for each $t \in [t_0, \infty)$ we obtain

$$y(t) = \lambda \left[y(t_0)e^{-\beta(t-t_0)} + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} e^{-\beta(t-s)} f(s, y_s) ds \right].$$

By (H_3), we have that for all $\theta \in [-h, 0]$ and $t \geqslant t_0 + h$,

$$\begin{aligned} |y(t+\theta)| &\leqslant |y(t_0)|e^{-\beta(t+\theta-t_0)} + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t+\theta} (t+\theta-s)^{\alpha-1} e^{-\beta(t+\theta-s)} |f(s, y_s)| ds \\ &\leqslant |y(t_0)|e^{-\beta(t+\theta-t_0)} + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t+\theta} (t+\theta-s)^{\alpha-1} e^{-\beta(t+\theta-s)} \left(K_1 + K_2\|y_s\|\right) ds, \end{aligned}$$

and thus

$$\|y_t\| \leqslant |y(t_0)|e^{-\beta(t-h-t_0)} + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} e^{-\beta(t-h-s)} \left(K_1 + K_2\|y_s\|\right) ds.$$

It follows from the arguments in (3.12)-(3.13), we can conclude that for each $t \in [t_0, \infty)$,

$$\frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} e^{-\beta(t-s)} ds \leqslant \frac{(t_0+h)^{\alpha-1} e^{-\beta(t_0+h)}}{\beta\Gamma(\alpha)} + \frac{(t_0+h)^{\alpha}}{\Gamma(\alpha+1)} =: R_1.$$

Hence

$$e^{\beta t}\|y_t\| \leqslant \|\phi\|e^{\beta(h+t_0)} + e^{\beta h}K_1R_1 + \frac{K_2e^{\beta h}}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}e^{\beta s}\|y_s\|ds.$$

Let $R_2 = \|\phi\|e^{\beta(h+t_0)} + e^{\beta h}K_1R_1$. Then from Lemma 2.2, there exists K such that we have for all $t \geqslant t_0 + h$,

$$\|y_t\| \leqslant R_2 + \frac{KK_2R_2e^{\beta h}}{\Gamma(\alpha+1)}(t-t_0)^{\alpha}e^{-\beta t}.$$

Since $\lim_{t\to\infty}(t-t_0)^{\alpha}e^{-\beta t} = 0$, there exists $R_3 > 0$ such that

$$\|y\|_{\infty} \leqslant R_3.$$

Set

$$U = \{y \in BC \ : \ ||y||_{\infty} < R_3 + 1\}.$$

$P : U \to BC$ is continuous and completely continuous. From the choice of U, there is no $y \in \partial U$ such that $y = \lambda P(y)$, for $\lambda \in (0,1)$. As a consequence of Leray-Schauder fixed point theorem, we deduce that P has a fixed point y in U.

Step 6. Uniform asymptotic stability of solutions.

Let $B \subset C([-h,0],\mathbb{R})$ be bounded, i.e., there exists $d \geqslant 0$ such that

$$\|\psi\| = \sup_{\theta\in[-h,0]}|\psi(\theta)| \leqslant d \quad \text{for all } \psi \in B.$$

From the similar arguments in step 4, we can deduce that there exists $R_4 > 0$ such that for all solutions $y(t,t_0,\phi)$ of IVP (1.1) with initial data $\phi \in B$, we have

$$\|y\|_{\infty} \leqslant R_4, \quad \forall \phi \in B.$$

Now we consider two solutions $x = x(t)$ and $y = y(t)$ of IVP (1.1) corresponding to initial

values ψ and ϕ. Note that for all $t \geqslant t_0$,

$$|x(t)-y(t)| \leqslant |x(t_0)-y(t_0)|e^{-\beta(t-t_0)} + \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}e^{-\beta(t-s)}\left(|f(s,x_s)|+|f(s,y_s)|\right)ds$$
$$\leqslant 2de^{-\beta(t-t_0)} + \frac{2}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}e^{-\beta(t-s)}\left(k_1(s)+k_2(s)R_4\right)ds. \tag{3.21}$$

Then the proof of uniform asymptotic stability of solutions can be done by making use of (H_4) and (3.21).

The proof of theorem 3.2 is completed. □

4 Examples

Example 4.1. Consider the fractional functional differential equation

$$\begin{cases} D^{\frac{1}{2}}\left[y(t)e^t\right] = \dfrac{e^{2t}}{8(e^t+e^{-t})}\sin^4\left(y\left(t-1\right)\right)+e^t, \quad t \geqslant 0, \\ y(t)=\phi(t), \quad -1 \leqslant t \leqslant 0, \end{cases} \tag{4.1}$$

where $f(t,y_t) = \frac{e^t}{8(e^t+e^{-t})}\sin^4\left(y\left(t-1\right)\right)+1$. It is clear that condition (H_0) holds. Let $x_t,\ y_t \in C([-1,0],\mathbb{R})$. Then for all $t \in [0,\infty)$, we have

$$\begin{aligned} |f(t,x_t)-f(t,y_t)| &= \frac{e^t}{8(e^t+e^{-t})}\left|\sin^4\left(x\left(t-1\right)\right)-\sin^4\left(y\left(t-1\right)\right)\right| \\ &\leqslant \frac{e^t}{2(e^t+e^{-t})}|x(t-1)-y(t-1)| \leqslant \frac{1}{2}|x(t-1)-y(t-1)|. \end{aligned}$$

On the other hand, note that $f(t,0)=1$ for each $t \in [0,\infty)$ and $\frac{1}{2}\left(\frac{e^{-1}}{\Gamma(\frac{1}{2})}+\frac{1}{\Gamma(\frac{3}{2})}\right) < 1$. Hence conditions (H_1) and (3.2) hold. By Theorem 3.1, we conclude that IVP (4.1) has a unique solution in the space $BC([-1,\infty),\mathbb{R})$, and the solutions of IVP (4.1) are uniformly asymptotically stable.

Example 4.2. Consider the fractional functional differential equation

$$\begin{cases} D^{\frac{1}{2}}\left[y(t)e^{t}\right] = 10e^{t}(t+1)^{-\frac{3}{4}}\dfrac{y(t-1)}{1+|y(t-1)|}, \quad t \geqslant 0, \\ y(t) = \phi(t), \quad -1 \leqslant t \leqslant 0, \end{cases} \tag{4.2}$$

where $f(t, y_t) = 10(t+1)^{-\frac{3}{4}}\frac{y(t-1)}{1+|y(t-1)|}$. It is easy to see that condition (H_2) holds. Let $y_t \in C([-1,0],\mathbb{R})$. Then for all $t \in [0,\infty)$, we find that

$$|f(t, y_t)| = \left|10(t+1)^{-\frac{3}{4}}\frac{y(t-1)}{1+|y(t-1)|}\right| \leqslant 10(t+1)^{-\frac{3}{4}}|y(t-1)|,$$

where $10(t+1)^{-\frac{3}{4}} \in BC([0,\infty),\mathbb{R}_+)$ with $\sup_{t\geqslant 0} 10(t+1)^{-\frac{3}{4}} = 10$, and

$$\frac{1}{\Gamma(\frac{1}{2})}\int_0^t (t-s)^{-\frac{1}{2}}e^{-(t-s)}10(s+1)^{-\frac{3}{4}}ds \leqslant \frac{10}{\Gamma(\frac{1}{2})}\int_0^t (t-s)^{-\frac{1}{2}}s^{-\frac{3}{4}}ds = \frac{10\Gamma(\frac{1}{4})}{\Gamma(\frac{3}{4})}t^{-\frac{1}{4}} \to 0 \text{ as } t \to \infty.$$

Thus conditions (H_3) and (H_4) hold, and the global existence and the uniform asymptotic stability of solutions of IVP (4.2) can be obtained by applying Theorem 3.2.

By using the algorithm given in [7], we numerically simulate Example 1 with the initial conditions $\phi(t) = \sin(t), \cos(t), -\cos(t), 1.5$, and Example 2 with $\phi(t) = t, \cos(t), -\cos(t), 1.5$, see Figures 1 and 2. From the numerical results, it can be noted that both of the solutions of Examples 1 and 2 converge uniformly, and the solutions of Example 1 converge faster than the ones of Example 2. The numerical results confirm the theoretical analysis.

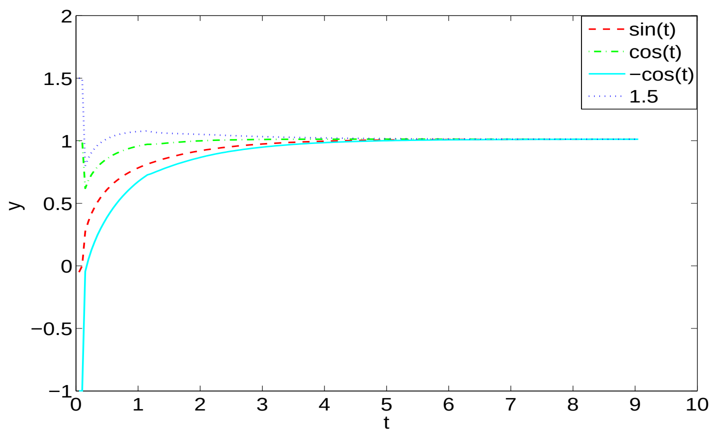

Figure 1: The numerical solutions of Example 1 with the initial conditions $\phi(t) = \sin(t), \cos(t), -\cos(t), 1.5$, respectively.

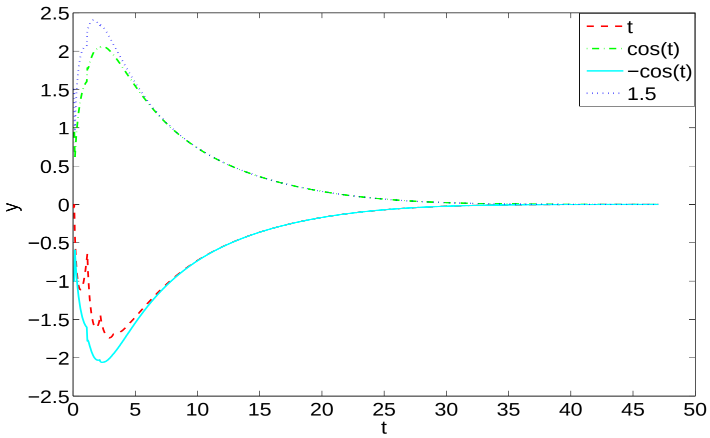

Figure 2: The numerical solutions of Example 2 with the initial conditions $\phi(t) = t, \cos(t), -\cos(t), 1.5$, respectively.

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